Inverse Problems for Neuronal Cables on Graphs

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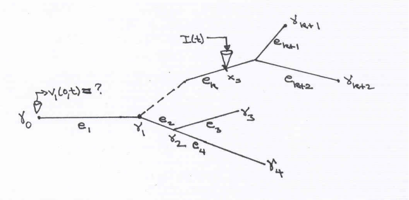
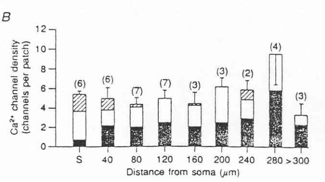
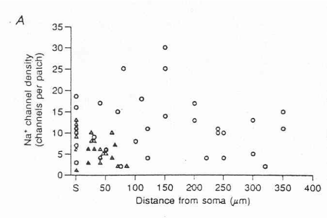
Motivation



Can we use boundary data (voltage & current measurements) to recover various distributed parameters in a dendritic tree model (quantum tree graph)?





Single Branch, Single Conductance

1. JB, G Cracian, 2005: numerical method on
   - single unknown ion density: $(1+q(x))u_t + q(x)u = u_{xx}$, $0 < x < L$, $u_x(0,t) = f(t), u_x(L,t) = 0$
   - unknown spine density in (nonlinear) Baer-Rinzel model:

$$
\begin{aligned}
u_t + u &= u_{xx} + \rho n(x)(v-u) \\
v_t + i_{HH}(v,m,n,h) &= \rho(u-v)
\end{aligned}
$$

2. D Wang, PhD thesis, UMBC, 2008: PDE-constrained
   optimization methods
   unknown spine density in Baer-Rinzel model

3. S Avdonin, JB, 2013, 2015 Boundary Control Theory on single unknown conductance $u_t + g(x)u = u_{xx}$

Dendritic Trees

R.C. Cannon et al. / Journal of Neuroscience Methods 84 (1998) 49–54 53

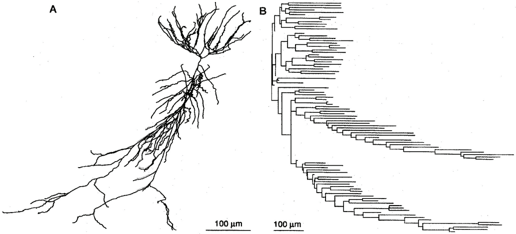
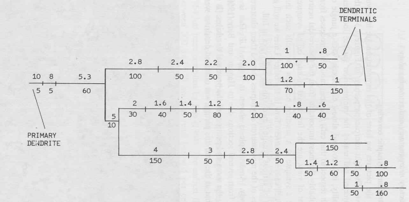


Fig. 3. A sample of representations available with the morphology editor as applied to the CA1 cell n184. A: a simple skeleton projection. B: the appropriate lengths and connections of all branches without direction information, in the form of a dendrogram.

motoneuron.jpg



Metric Graphs

Let $\Gamma = \{E, V\}$ be a finite compact metric graph.

$E = \{e_j\}_{j=1}^N$ is a set of edges and $V = \{\nu_j\}_{j=1}^M$ is a set of vertices. A graph is called a metric graph if every edge $e_j \in E$ is identified with an interval (a_{2j-1}, a_{2j}) of the real line with a positive length $l_j = |a_{2j-1} - a_{2j}|$, and a graph is a tree if it has no cycles. The edges are connected at the vertices v_j which can be considered as equivalence classes of the edge end points $\{a_j\}$.
Let $\{\gamma_1, \ldots, \gamma_m\} = \partial\Gamma \subset V$ be the boundary vertices.

In this talk, graph = (finite compact) metric tree graph.

A Simple Tree Example

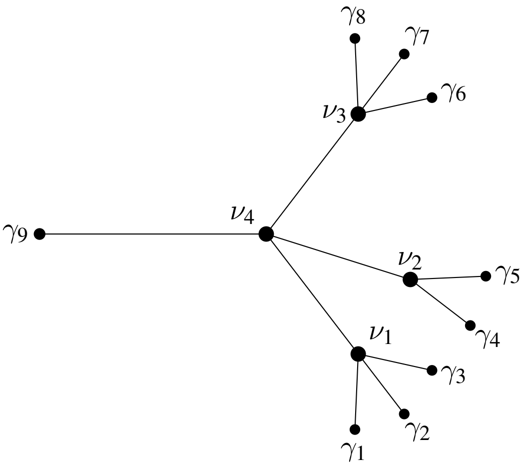


Fig. 1: A metric tree with $m = 9$ and $N = 12$

Quantum Graphs

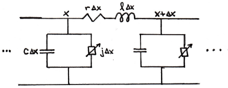
Quantum graph $\{\Gamma, H\}$: **differential operator H on metric graph Γ, coupled by specific vertex matching conditions.**

Applications include

- Oscillations of the flexible structures made of strings, beams, cables.
- hierarchical materials like ceramic or metallic foams, percolation networks and carbon and graphene nano-tubes.
- Our motivation comes from neurobiology; specifically dendritic trees of CNS neurons.

Control and inverse theories for PDEs on graphs constitute an important part of the rapidly developing area of applied mathematics — analysis on graphs.

Neuronal Cable Equation



Single dendritic branch ⇒ Circuit model ⇒ Cable model
Non-dimensionalized neuronal cable equation has the form

$$v_t + \sum_{j=1}^{k} g_j(x)(v - E_j) = v_{xx}$$

Pick any $1 \leq i \leq k$, and let $u = v - E_i$, $E_{ji} = E_j - E_i$, and substitute into the equation:

$$u_t - u_{xx} + q(x)u = h(x)$$

where $q(x) = \sum_{j=1}^{k} g_j(x)$, and $h(x) = \sum_{j \neq i} E_{ji} g_j(x)$. The $E_j s$ are assumed known, but the $g_j s$, hence $q(x)$ and $h(x)$, are assumed unknown a priori.

IBVP on a Graph

Consider a system described by neuronal cable theory on a graph (adding $p(t)$, with $p(0) \neq 0$):

$$\partial_t u_j - \partial_x^2 u_j + q_j(x) u_j = p(t) h_j(x) \ \text{ on } \ e_j \times (0,T) \ \forall e_j \in E \tag{1}$$

or shortly, $\partial_t u - \partial_x^2 u + q(x)u = p(t)h(x)$ on $E \times (0,T)$

KN: $\sum_{e_j \sim \nu} \partial u_j(\nu, t) = 0$ at each vertex $\nu \in V \setminus \partial\Gamma$,
$u(\cdot, t)$ is continuous at each vertex, for $t \in [0,T]$ (2)

$$\partial u = f \ \text{ on } \ \partial\Gamma \times [0,T], \ \ u|_{t=0} = 0 \ \text{ on } \ E. \tag{3}$$

In (2) (and below) $\partial u_j(\nu)$ denotes the derivative of u at the vertex ν taken along the edge e_j in the direction outwards the vertex. Also, $e_j \sim \nu$ means edge e_j is incident to vertex ν, and the sum is taken over all edges incident to ν.

Let $\mathcal{H} = L^2(\Gamma)$ and $\mathcal{F}^T = L^2([0,T];\mathbb{R}^m)$.

Theorem 1. If $f, p \in \mathcal{F}^T$, $h \in \mathcal{H}$, then for any $t \in [0,T]$, $u^f(\cdot,t) \in \mathcal{H}$ and $u^f \in C([0,T];\mathcal{H})$.

For the inverse problem purposes we assume that $p \in H^1(0,T)$, $p(0) \neq 0$.
The response operator, $R^T : \mathcal{F}^T \to \mathcal{F}^T$, is defined as

$$(R^T f)(t) = u^f(\cdot,t)|_{\partial\Gamma}, \ t \in [0,T].$$

Theorem 2. Operator R^T known for any $T > 0$ uniquely determines the graph topology, the lengths of the edges, the potentials q_j and sources h_j, $j = 1, \dots, N$.

Separating Problem into Two Auxiliary Problems

Solution of the problem (1)–(3) can be presented in the form $u = y + z$:

$$\partial_t y - \partial_x^2 y + q(x)y = 0 \text{ on } E \times (0, T) \tag{4}$$

$$\partial y = f \quad \text{on} \quad \partial\Gamma \times [0, T] \tag{5}$$

$$\partial_t z - \partial_x^2 z + q(x)z = p(t)h(x) \text{ on } E \times (0, T) \tag{6}$$

$$\partial z = 0 \quad \text{on} \quad \partial\Gamma \times [0, T] \tag{7}$$

with the KN matching conditions and zero initial conditions. We see that $z = u^0$ so, $y^f|_{\partial\Gamma} = u^f|_{\partial\Gamma} - u^0|_{\partial\Gamma} = R^T f - R^T 0.$

Solving Problem 1: Main Ideas

$\mathcal{L}\phi \doteq -\phi'' + q(x)\phi = \lambda\phi$ on $\Gamma \setminus V$, KN at $V \setminus \partial\Gamma$, $\phi'(\gamma_j, \lambda) = f_j$,
$f = col(f_1, \ldots, f_m)$, $\rightarrow\ \exists!\ \phi = \phi^f$

1. BC method allows spectral data SD=$\{\lambda_n, \phi^f_{n\ |\partial\Gamma}\}_{n\in\mathbb{N}}$ to be recovered from R^T
2. SD determines TW matrix function M: $\phi^f_{|\partial\Gamma} = M(\lambda)f$
3. TW matrix function M used in reduction method

Essence: recalculate M from original graph to smaller graph by 'pruning' the boundary edges. By iterative process we reduce original IP (Prob. 1) to IP for **companion wave problem** on single edge.

$w_{tt} - w_{xx} + q(x)w = 0$ on $e = (0, \ell)$, zero i.c.s,
$w_x(0, t) = f(t), w_x(\ell, t) = 0$,
$f \in \mathcal{H}^T \doteq \{f \in L^2(0, \ell) \mid supp\, f \subset [0, T]\}$

Problem 1 continued

$F(t) = \int_0^t f(s)ds \rightarrow$
$w^f(x,t) = -F(t-x) + \int_x^t \kappa(x,s)F(t-s)ds,\ x < t,\ w^f(x,t) = 0, x \geq t$

$\kappa(x,t) =$ sol. to a certain Goursat prob. on $0 < x < t < T \leq \ell$,
$r(t) = \kappa(0,t)$

$\psi'' - q(x)\psi = 0,\ \psi(0) = 1, \psi'(0) = 0, \quad q \in L^1(0,\ell)$

$\exists f = f^T \in L^2(0,T)$ s.t. $w_t^{f^T}(x,T) = \psi(x), x \leq T, = 0, x > T$

$f^T(t) - \frac{1}{2}\int_0^T \{r(|t-s|) + r(2T-t-s)\}f(s)ds = -1 \quad t \in [0,T]$

$w_t^{f^T}(T-,T) = -f^T(0+) = \psi(T) \in C^2 \Rightarrow q(T) = \frac{\psi''(T)}{\psi(T)}$

See **Avdonin and Bell, J. Inv. Prob. and Imaging, 9(3)(2015)**

Solving Problem 2

$$z(x,t) = \sum_{n=1}^{\infty} \varphi_n(x)\, h_n \int_0^t p(t-\tau) e^{-\lambda_n \tau}\, d\tau \ ,$$

where $h_n = \langle h, \varphi_n \rangle_{\mathcal{H}}$. Recalling $p(0) \neq 0$, $\mu(t) \doteq u^0(\cdot, t)|_{\partial\Gamma}$

$$\mu(t) = \int_0^t p(t-\tau) W(\tau)\, d\tau, \quad W(\tau) \doteq \sum_{n=1}^{\infty} h_n \varphi_n(0) e^{-\lambda_n \tau},$$

$$\mu'(t) = p(0)\, W(t) + \int_0^t p'(t-\tau) W(\tau)\, d\tau.$$

The last equation allows finding $W(t)$ as the solution to the VIE.

Problem 2 continued

Theorem 3 (controllability): implies that the family $\{\varphi_n(0)e^{-\lambda_n t}\}$ is minimal in $\mathcal{F}^T = L^2([0,T];\mathbb{R}^m)$ (for any $T > 0$) with biorthogonal family $\{\theta_n(t)\}$.
Thus, with $W(\cdot)$ and SD determined, we then have the $h_n s$ and h:

$$h_n = \langle W, \theta_n \rangle_{\mathcal{F}^T}, \quad h(x) = \sum_{n=1}^{\infty} h_n \varphi_n(x).$$

Recovery of original distributed parameters requires solving Problem 1 once, solving Problem 2 k-1 times.

See **Avdonin, Bell, and Nurtazina, Math. Meth. Appl. Sci. 40(2017)**

Other Problems

Scaled (linear) models:

- One unknown ion species: $(1 + q(x))u_t + q(x)u = u_{xx}$
  $(C_m = C_0 + C_1 N(x), g = g_1 N(x))$
- Unknown radius, known conductance:
  $u_t + q(x)u = \frac{1}{a(x)}(a(x)^2 u_x)_x$

First on single interval, then on a graph.

Next: Graphs with Cycles

cyclomatic number (first Betti number) $= |E| - |V| + 1 (> 0)$
Much more challenging extending control theory than for tree graphs

network.jpg