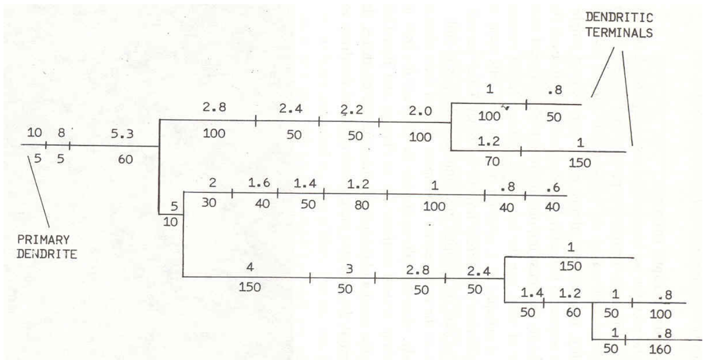
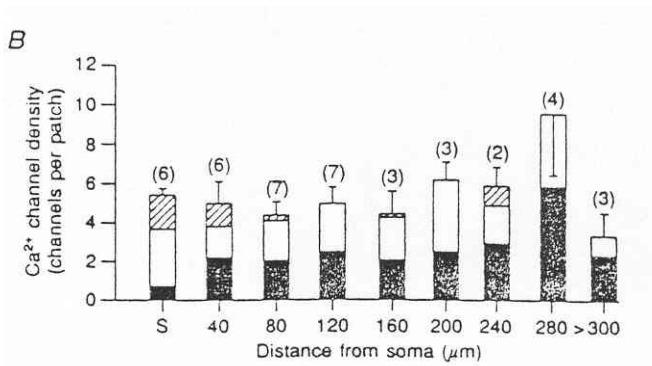
Introduction to the Boundary Control Method and its Application to Inverse Problems

By Jon Bell

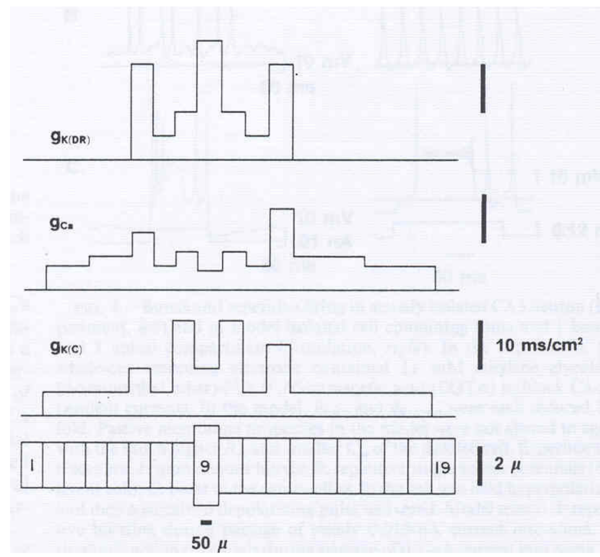


Cat motoneuron (J. Barrett, 1988)

Can we use boundary data, that is, voltage and current measurements (with a recording microelectrode) to recover various distributed parameters?



McGee and Johnston, J. Neurophys., 1989

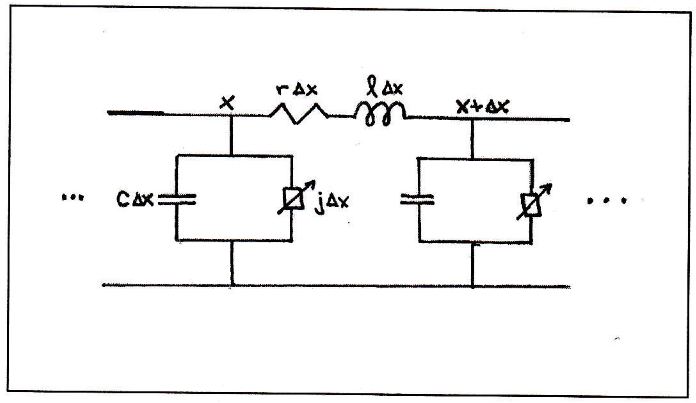


Traub, J. Physiol. 1995

Outline:

- Statement of a problem to address on a graph and its ‘reduction’
- Problem on a branch and some background
- Companion (wave) problem and Neumann condition: development of the Boundary Control Method
- Problems being worked on and a ‘holy grail’ problem

Introduction to Cable Theory



c membrane capacity / length

ℓ inductance / length

r resistance / length

$$\boxed{\begin{aligned} &\frac{\partial i}{\partial x} + c(x)\frac{\partial v}{\partial t} + j(v) = 0 \\ &\frac{\partial v}{\partial x} + \ell(x)\frac{\partial i}{\partial t} + r(x)\, i = 0 \end{aligned}}$$

$c, \ell, r > 0$ constants

$j(v) =$ current-voltage relation

Eliminate $i \Rightarrow$

$$\ell c\, v_{tt} + (rc + \ell j'(v))\, v_t + r j(v) = v_{xx}$$

Nondimensionalize : $\tilde{v} = v/E \quad \tilde{x} = \sqrt{r g_0}\, x, \quad \tilde{t} = t g_0 / c$

$$\tilde{J}(\tilde{v}) = j(E\tilde{v})/g_0 E$$

substitute these, divide by $r g_0 E$, drop tilde notation:

$$\varepsilon v_{tt} + (1 + \varepsilon J'(v))\, v_t + J(v) = v_{xx}$$

$\varepsilon := \ell g_0 / rc$. For a linear (passive) cable, $j(v) = g_0 q(x)\, v$.

In Neuronal cable theory $\Leftrightarrow$ no inductance, $\ell \to 0$ $(\varepsilon \to 0)$:

$$v_t + J(v) = v_{xx}$$

Linear case : $\quad v_t + q(x)\, v = v_{xx}$

Passive Cable Theory with (unknown) Distributed Conductance on a Metric Tree Graph

$\Omega = E \oplus V = \{e_1, \dots, e_N\} \oplus \{v_1, \dots, v_M\}$ is a finite metric tree graph

A <u>metric graph</u>: every edge $e_j \in E$ is identified with an interval of the real line of length $l_j \in \mathbb{R}^+$

A <u>tree graph</u> has no cycles

$\partial\Omega = \{v \in V : \text{index}(v) = 1\}$; $v \in V \setminus \partial\Omega \rightarrow \text{index}(v) \geq 3$

$= \{\gamma_1, \dots, \gamma_m\}$

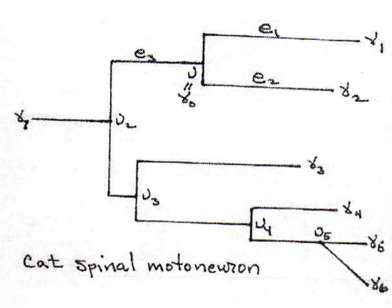
(1) $v_t + q(x)\, v = v_{xx}$ in $\{\Omega \setminus V\} \times (0, T)$

(2) KN: $\displaystyle\sum_{e_j \sim v} \partial v_j(v, t) = 0$ $\quad v \in V \setminus \partial\Omega$, $t \in [0, T]$

$v(\cdot, t)$ continuous at each $v \in V$, $t \in [0, T]$

(3) $\partial v = f$ on $\partial\Omega \times [0, T]$

(4) $v|_{t=0} = 0$ in Ω



Cat spinal motoneuron

<u>IDea</u>: Determine conductance on e_1, e_2 via boundary data at γ_1, γ_2, KN at v ; do this for all boundary edges. Then 'prune' tree of these boundary edges for a smaller tree, and iterate process.

So we need a procedure to solve the problem on a single edge.

Unknown Distributed Parameter on a Single Branch

$$\begin{cases} v_t + q(x)\, v = v_{xx} & 0 < x < L,\ 0 < t < T \\ v(x,0) = 0 & 0 < x < L \\ v_x(0,t) = f(t), \quad v_x(L,t) = 0 & 0 < t < T \end{cases}$$

Some Work

1. Pure numerical method (B, Craciun, 2005)

   single ion density: $\boxed{(1+q(x))\, v_t + q(x) v = v_{xx}}$

   Baer-Rinzel model of dendrite with spines:

   nonlinear dynamics

   $n(x)$ = unknown spine density

2. PDE - Constrained optimization method

   (D. Wang, PhD dissertation, UMBC, 2008)

   Baer-Rinzel model

3. Boundary Control Method (BCM) (Avdonin-B, 2012)

$$
(D) \quad \begin{cases} u_t + q(x) u = u_{xx} & 0<x<L, \quad 0<t<T \\ u(x,0) = 0 & \\ u_x(0,t) = f(t), \quad u_x(L,t) = 0 & f \in \mathcal{Y}^T := L^2(0,T) \end{cases}
$$

response operator: $R^T : \mathcal{Y}^T \to \mathcal{Y}^T$, $(R^T f)(t) = u^f(0,t)$

(Neumann-to-Dirichlet map of the problem)

dynamic IP: recover unknown conductance $q(x)$ from R^T

Define $\mathcal{L}$ by $\quad (\mathcal{L}\varphi)(x) = -\frac{d^2\varphi}{dx^2}(x) + q(x)\varphi(x) \quad x \in (0,L)$

$\mathrm{dom}[\mathcal{L}] = \{ \varphi \in H^2(0,L) : \varphi'(0) = \varphi'(L) = 0 \}$

$\mathcal{L}$ is self-adjoint in $\mathcal{H} \doteq L^2(0,L)$; as an operator with compact resolvent has only a discrete spectrum $\{\lambda_n\}$ with ON set of eigenfunctions $\{\varphi_n\}$.

Spectral Data (SD) $= \{\lambda_n, \varphi_n(0)\}_{n\in\mathbb{N}} = \{\lambda_n, \kappa_n\}_{n\in\mathbb{N}}$

$$
u(x,t) = u^f(x,t) = \sum_n a_n(t)\varphi_n(x)
$$

So $\quad (R^T f)(t) = \int_0^t r(t-s) f(s)\, ds \qquad t \in [0,T]$

$$
r(t) \doteq -\sum_1^\infty \kappa_n^2 e^{-\lambda_n t} \quad = u^\delta(0,t)
$$

Companion Wave Problem

$$
(W) \quad \begin{cases} w_{tt} + q(x) w = w_{xx} & 0<x<L, \quad 0<t<T \\ w(x,0) = w_t(x,0) = 0 & 0<x<L \\ w_x(0,t) = f(t), \quad w_x(L,t) = 0 & 0 \le t \le T \end{cases}
$$

Problems (D) and (W) share the same EVP and (W) has 'more tractable' dynamics

To establish an algorithmic approximation of the conductance in problem (D) we require high-order accuracy of the SD for a sufficient number of eigenvalues — a significant challenge

plus tractability of the IP for problem (W).

Therefore, in this talk we will just concentrate on the recovery of the "potential" for problem (W).

<u>Remark</u> : there is significant literature on the Dirichlet analogue to problem (W)

(so $u(0,t) = f(t)$, $u(L,t) = 0$)

That problem is associated with <u>quantum wires</u>.

(Belishev, 2007 ; Avdonin, 2008 ; Avdonin, Belinskiy, Matthews, 2011 ; etc.)

The (W) Problem and the Boundary Control Method

$$(W)\quad \begin{cases} w_{tt} + q(x)w = w_{xx} & x>0, \quad 0<t<T, \quad \text{any } T>0 \\ w(x,0) = w_t(x,0) = 0 & x>0 \\ w_x(0,t) = f(t) & \end{cases}$$

Theory works for $f \in \mathcal{F}^T \doteq L^2(0,T)$, $q \in L^1(0,\infty)$ leading to generalized solutions

For this talk, assume $f, q \in C^\infty[0,\infty)$, which leads to classical solutions (given compatibility condition on f)

Let $F(t) = \int_0^t f(s)\,ds$

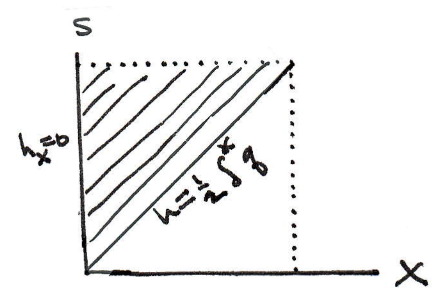
Lemma 1*: (W) has a unique solution $w = w^f(x,t)$ given by

$$(1)\quad w^f(x,t) = \begin{cases} -F(t-x) + \int_x^t h(x,s)\,F(t-s)\,ds & \text{for } x<t \\ 0 & x \geq t \end{cases}$$

Here h is the solution to the following Goursat problem:

Lemma 2: For $q \in L^1(0,\infty)$, h solves

$$(2)\quad \begin{cases} h_{ss} - h_{xx} + q(x)h = 0 & 0<x<s \\ h_x(0,s) = 0, \quad h(x,x) = \frac{1}{2}\int_0^x q & \end{cases}$$



(If one assumes a solution form for w^f like (1) and substitutes it into (W), then (2) comes out. Lemma 2 is classical.)

* Similar to arguments in Avdonin, Mikhaylov, Rybkin, 2007.

Let $\xi = s+x$, $\eta = s-x$ $\quad h(x,s) = V(s+x, s-x) = V(\xi,\eta)$

Then $h_s = V_\xi + V_\eta$, $h_x = V_\xi - V_\eta$, etc. $\Rightarrow$

$$(3) \quad \begin{cases} V_{\xi\eta} + \frac{1}{4} q\left(\frac{\xi-\eta}{2}\right) V = 0 \\ V_\xi(\xi,\xi) = V_\eta(\xi,\xi), \quad V(\xi,0) = \frac{1}{2}\int_0^{\xi/2} q(s)\,ds \end{cases}$$

Thus,

$$V_\xi(\xi,\eta) = V_\xi(\xi,0) - \frac{1}{4}\int_0^\eta q\left(\frac{\xi-\eta_1}{2}\right) V(\xi,\eta_1)\,d\eta_1$$

$$(4) \quad = \frac{1}{4} q(\xi/2) - \frac{1}{4}\int_0^\eta q\left(\frac{\xi-\eta_1}{2}\right) V(\xi,\eta_1)\,d\eta_1$$

Response operator : $R^T : \mathcal{Y}^T \to \mathcal{Y}^T$ given by $(R^T f)(t) = w_t^f(0,t)$

From (4)

$$(5) \quad (R^T f)(t) = -f(t) + \int_0^t r(s) f(t-s)\,ds \quad \text{, where}$$

$$(6) \quad r(s) = h(0,s) \quad \text{(response function)}$$

<u>Remark</u> : The "forward problem" is to obtain r from q.

$$r'(s) = h_s(0,s) = V_\xi(s,s) + V_\eta(s,s) = 2V_\xi(s,s)$$

$$= \frac{1}{2} q(s/2) - \frac{1}{2}\int_0^s q\left(\frac{s-\eta_1}{2}\right) V(s,\eta_1)\,d\eta_1 \quad \text{, or}$$

$$(7) \quad r(t) = \frac{1}{2}\int_0^s q(s'/2)\,ds' - \frac{1}{2}\int_0^s \int_0^{s'} q\left(\frac{s'-\eta_1}{2}\right) V(s',\eta_1)\,d\eta_1\,ds'$$

Integrating (4),

$$V(\xi,\eta) = \underbrace{V(\eta,\eta)}_{=r(\eta)} + \frac{1}{2}\int_{\eta/2}^{\xi/2} q(\xi_1)\,d\xi_1 - \frac{1}{4}\int_\eta^\xi \int_0^\eta q\left(\frac{\xi_1-\eta_1}{2}\right) V(\xi_1,\eta_1)\,d\eta_1\,d\xi_1$$

$\Rightarrow$ (substituting (7)

(8) $$V(\xi,\eta)=\frac{1}{2}\int_0^{\xi/2} q(s)\,ds+\frac{1}{4}\int_0^{\eta/2} q(s)\,ds-\frac{1}{4}\int_\eta^{\xi}\int_0^{\eta} q\left(\frac{\xi_1-\eta_1}{2}\right)V(\xi_1,\eta_1)\,d\eta_1\,d\xi_1$$
$$-\frac{1}{2}\int_0^{\eta}\int_0^{\xi_1} q\left(\frac{\xi_1-\eta_1}{2}\right)V(\xi_1,\eta_1)\,d\eta_1\,d\xi_1$$

(A Picard-type iterative approach to this IE converges to a solution to (3), hence getting Lemma 2)

Boundary Control Method

Connection operator : $C^T : \mathcal{Y}^T \to \mathcal{Y}^T$ defined by bilinear form

(9) $$(C^T f, g)_{\mathcal{Y}^T} = (w_t^f(\cdot,T), w_t^g(\cdot,T))_{L^2}$$

This suggests control operator $U^T : \mathcal{Y}^T \to \mathcal{Y}^T$, $(U^T f) = w_t^f(\cdot,T)$

$\Rightarrow C^T = (U^T)^* U^T$, $(U^T)^*$ = adjoint of U^T

U^T is a bounded op. on $\mathcal{Y}^T \Rightarrow C^T$ is bounded op on $\mathcal{Y}^T$.

Consider IVP

(10) $$\begin{cases} -\dfrac{d^2y}{dx^2} + q(x)\,y = 0 & x > 0 \\ y(0) = 1 , \quad \dfrac{dy}{dx}(0) = 0 \end{cases}$$

<u>Lemma 3</u> (Exact Controllability): If q is known, then for solution y of (10), there exists a unique control $f = f^T \in \mathcal{Y}^T$ such that

(11) $$w_t^{f^T}(x,T) = \begin{cases} y(x) & \text{if } x \le T \\ 0 & x > T \end{cases}$$

Remark: At this stage q is unknown, so y & f^T are also unknown.

From Avdonin - B, 2012, for $g \in C_0^\infty[0,T]$

$$(C^T f^T, g) = (w_t^{f^T}(\cdot,T), w_t^g(\cdot,T)) = \int_0^T y(x) w_t^g(x,T)\,dx$$

$$= \int_0^T \int_0^T y(x) w_{tt}^g(x,t)\,dx\,dt = \int_0^T \int_0^T y(x)\left[w_{xx}^g - q w^g\right] dx\,dt$$

$$= -\int_0^T y(0) w_x^g(0,t)\,dt = -\int_0^T g(t)\,dt$$

so f^T satisfies

(12) $\quad (C^T f^T)(t) = -1 \qquad t \in [0,T]$

<u>Lemma 4</u>: If $f \in \mathcal{F}^T$, the following representation is valid:

(13) $\quad (C^T f)(t) = f(t) - \frac{1}{2}\int_0^T \{r(|t-s|) + r(2T-t-s)\} f(s)\,ds, \quad t \in [0,T]$

(Trick is to let $f, g \in C_0^\infty[0,T]$, let $f_-(t) = \begin{cases} f(t) & 0 \le t \le T \\ f(2T-t) & T < t \le 2T \end{cases}$

Define $W(s,t) = (u_s^f(\cdot,s), u_t^g(\cdot,t))_{\mathcal{H}^T}$ $\quad$ (so $W(T,T) = (C^T f, g)_{\mathcal{F}^T}$)

A tedious calculation shows $W(s,t)$ satisfies

$$\begin{cases} W_{tt} - W_{ss} = f_-'(s)(R^T g)(t) - (R^{2T} f_-)(s)\, g'(t) \\ W(s,0) = 0, \quad W_t(s,0) = 0 \end{cases}$$

so by d'Alembert on domain (s,t) $\quad 0 \le s \le T, \; s \le t \le 2T-s$

$$W(T,T) = \frac{1}{2}\int_0^T \int_s^{2T-s} \{f_-'(t)(R^T g)(s) - g'(s)(R^{2T} f_-)(t)\}\,dt\,ds$$

(13) is obtained from this from a number of transformations, and the definition of f_-.

From (12), (13)

$$(14) \qquad f^T(t) - \frac{1}{2}\int_0^T \{r(|t-s|) + r(2T-t-s)\} f^T(s)\,ds = -1 \qquad t \in [0,T]$$

From (1) (the representation of w^f in terms of h)

$$w_t^{f^T}(x,t) = -f^T(t-x) + \int_x^t h(x,s) f^T(t-s)\,ds \qquad x < t$$

so for $t = T$, take limit to obtain

$$w_t^{f^T}(T-,T) = -f^T(0+)$$

Therefore, from (11) (the exact controllability lemma)

$$(15) \qquad y(T) = w_t^{f^T}(T-,T) = -f^T(0+)$$

and from the IVP, (10),

$$(16) \qquad q(T) = \frac{y''(T)}{y(T)}.$$

Summary :

$$(7) \quad r(s) = \frac{1}{2}\int_0^s q(s'/2)\,ds' - \frac{1}{2}\int_0^s\int_0^{s'} q\left(\frac{s'-\eta_1}{2}\right)V(s',\eta_1)\,d\eta_1\,ds'$$

$$(8) \quad V(\xi,\eta) = \frac{1}{2}\int_0^{\xi/2} q(s)\,ds + \frac{1}{2}\int_0^{\eta/2} q(s)\,ds - \frac{1}{4}\int_\eta^\xi\int_0^\eta q\left(\frac{\xi_1-\eta_1}{2}\right)V(\xi_1,\eta_1)\,d\eta_1\,d\xi_1$$

$$- \frac{1}{2}\int_0^\eta\int_0^{\xi_1} q\left(\frac{\xi_1-\eta_1}{2}\right)V(\xi_1,\eta_1)\,d\eta_1\,d\xi_1$$

$$(14) \quad f^T(t) - \frac{1}{2}\int_0^T \{r(|t-s|) + r(2T-t-s)\}\, f^T(s)\,ds = -1 \qquad t\in[0,T]$$

$$(15) \quad y(T) = -f^T(0+)$$

$$(16) \quad q(T) = \frac{y''(T)}{y(T)}$$

So, for the IP,

i) make an initial guess for q

ii) solve for r in (7), using (8)

iii) then obtain f^T from (14), marching out in $T>0$

iv) use $f^T(0)$ for $y(T)$ in (15)

v) use center differences in (16) to obtain a new approximation of q

iterate this scheme.

Companion Projects

1. Diffusion (D) problem, extending to tree graph, using Weyl-Titchmarsh matrix function theory (in progress)

2. Recovery of single ion channel model (single branch):

$$\begin{cases} (1+q(x))\, u_t + q(x) u = u_{xx} & 0<x<L \quad 0<t<T \\ u(x,0) = 0 & 0<x<L \\ u_x(0,t) = f(t) \qquad u_x(L,t) = 0 & 0<t<T \end{cases}$$

(in progress)

Remark: the companion wave problem is

$$\begin{cases} \rho(x)\, u_{tt} - u_{xx} + q(x) u = 0 & 0<x<L,\ 0<t<T, \quad \rho \doteq 1+q \\ u(x,0) = u_t(x,0) = 0 & \\ u_x(0,t) = f(t) \qquad u_x(L,t) = 0 & \end{cases}$$

Let $\tau(x) = \int_0^x \sqrt{\rho(s)}\, ds \quad \rightarrow x(\tau)$ is its inverse

$$u(x,t) = \left(\frac{\rho(x(\tau))}{\rho(0)}\right)^{-1/4} \tilde{u}(\tau,t) \rightarrow$$

$$\begin{cases} \tilde{u}_{tt} - \tilde{u}_{\tau\tau} + \tilde{q}(\tau)\, \tilde{u} = 0 & 0<\tau<\tau(L) \quad 0<t<T \\ \tilde{u}(\tau,0) = \tilde{u}_t(\tau,0) = 0 & \\ \tilde{u}_\tau(0,t) - \beta\, \tilde{u}(0,t) = \tilde{f}(t) & \end{cases}$$

$$\tilde{f}(t) \doteq f(t)/\sqrt{\rho(0)}$$

$$\beta \doteq \frac{1}{4} \frac{\rho'(0)}{\rho(0)^{3/2}}$$

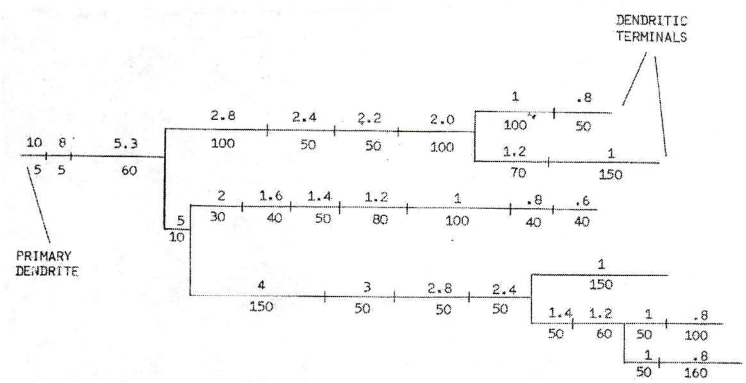
so former theory can be exploited.

A "Holy Grail" Problem

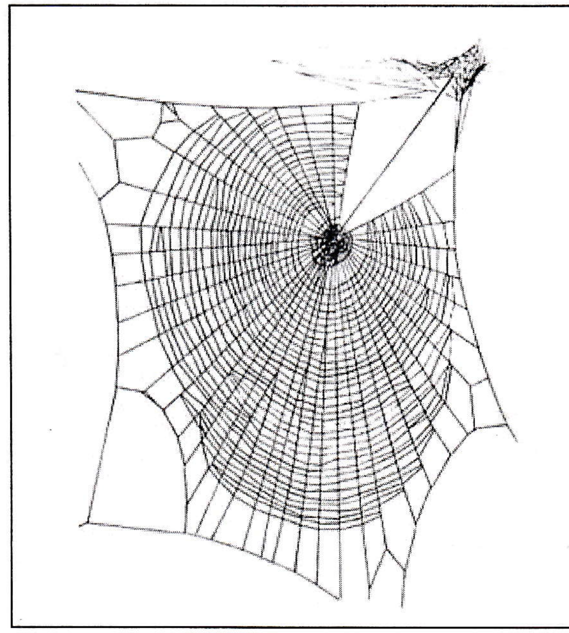
Belishev, 2004 : his planar graphs are metric tree graphs with densities on the edges. For the Dirichlet problem, the SD is the spectrum and derivatives of the normalized eigenfunctions at the boundary vertices.

He showed the graph topology and the densities of its edges are determined by the spectral data uniquely up to a natural isometry in the plane (by the BCM).

Can a similar result for the Diffusion-Neumann problem be obtained for a graph representing a dendritic tree?

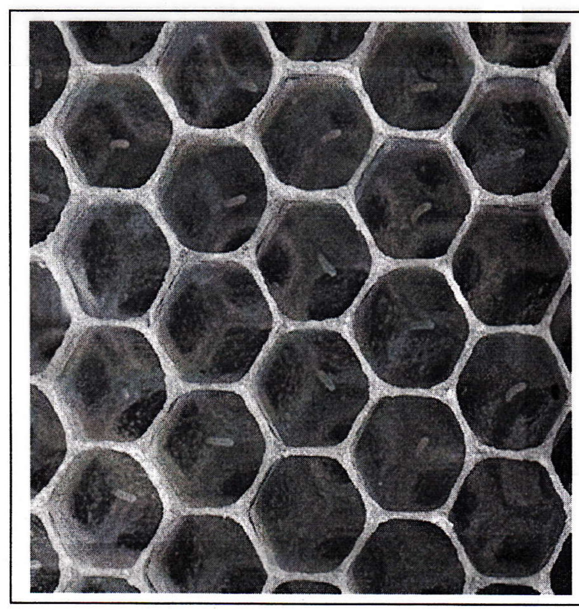


Landolfa, Barth, 1996; Eberhard, Chacón, 1980:



Webs are an extension of the spider's sensory space, as well as a mechanical device for prey interception and retention. The spider must solve two problems with the web, namely discrimination of and orientation towards detected sources. How does the web structure affect sensitivity and directional transmission?

Honeycombs are examples of <u>ramified</u> <u>spaces</u>



Sandeman, et al, 1996:

Vibration of the rims of open honeycomb cells is transmitted across the comb. Honeybees have receptors on their legs for detecting vibrations in the range emitted by a bee during its dance. Framed (commercial) combs strongly attenuate higher frequencies, whereas these frequencies are amplified in small, open (natural) combs. Bees have a habit of freeing an area of comb from the frame in those areas used for dancing.