Inverse Problems for Neuronal Cables on Graphs

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Metric Graphs

Let $\Gamma = \{E, V\}$ be a finite compact metric graph.

$E = \{e_j\}_{j=1}^N$ is a set of edges and $V = \{\nu_j\}_{j=1}^M$ is a set of vertices. A graph is called a metric graph if every edge $e_j \in E$ is identified with an interval (a_{2j-1}, a_{2j}) of the real line with a positive length $l_j = |a_{2j-1} - a_{2j}|$, and a graph is a tree if it has no cycles. The edges are connected at the vertices v_j which can be considered as equivalence classes of the edge end points $\{a_j\}$.
Let $\{\gamma_1, \ldots, \gamma_m\} = \partial\Gamma \subset V$ be the boundary vertices.

In this talk, graph = (finite compact) metric tree graph.

A Simple Tree Example

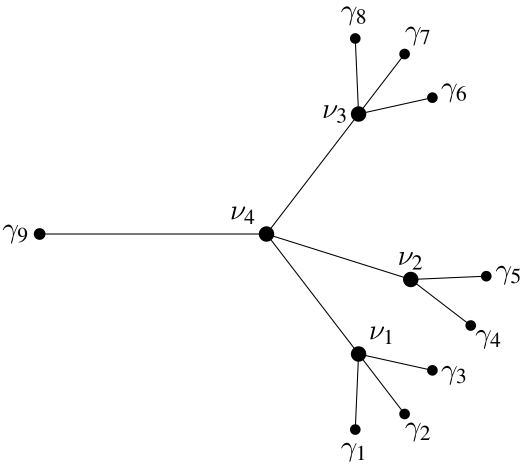

Fig. 1: A metric tree with $m = 9$ and $N = 12$

Quantum Graphs

Quantum graph $\{\Gamma, H\}$: differential operator H on metric graph Γ, coupled by specific vertex matching conditions.

Applications include

- Oscillations of the flexible structures made of strings, beams, cables.
- hierarchical materials like ceramic or metallic foams, percolation networks and carbon and graphene nano-tubes.
- Our motivation comes from neurobiology; specifically dendritic trees of CNS neurons.

Control and inverse theories for PDEs on graphs constitute an important part of the rapidly developing area of applied mathematics — analysis on graphs.

Neuronal Cable Equation

Non-dimensionalized neuronal cable equation has the form

$$v_t + \sum_{j=1}^{k} g_j(x)(v - E_j) = v_{xx}$$

Pick any $1 \le i \le k$, and let $u = v - E_i$, $E_{ji} = E_j - E_i$, and substitute into the equation:

$$u_t - u_{xx} + q(x)u = h(x)$$

where $q(x) = \sum_{j=1}^{k} g_j(x)$, and $h(x) = \sum_{j \neq i} E_{ji} g_j(x)$. The $E_j s$ are assumed known, but the $g_j s$, hence $q(x)$ and $h(x)$, are assumed unknown a priori.

IBVP on a Graph

Consider a system described by neuronal cable theory on a graph:

$$\partial_t u_j - \partial_x^2 u_j + q_j(x) u_j = p(t) h_j(x) \ \text{ on } \ e_j \times (0,T) \ \forall e_j \in E \tag{1}$$

or shortly, $\partial_t u - \partial_x^2 u + q(x) u = p(t) h(x)$ on $E \times (0,T)$

KN: $\sum_{e_j \sim \nu} \partial u_j(\nu, t) = 0$ at each vertex $\nu \in V \setminus \partial\Gamma$,
$u(\cdot, t)$ is continuous at each vertex, for $t \in [0,T]$ (2)

$$\partial u = f \ \text{ on } \ \partial\Gamma \times [0,T], \ \ u|_{t=0} = 0 \ \text{ on } \ E. \tag{3}$$

In (2) (and below) $\partial u_j(\nu)$ denotes the derivative of u at the vertex ν taken along the edge e_j in the direction outwards the vertex. Also, $e_j \sim \nu$ means edge e_j is incident to vertex ν, and the sum is taken over all edges incident to ν.

IBVP and Inverse Problem

Let $\mathcal{H} = L^2(\Gamma)$ and $\mathcal{F}^T = L^2([0,T];\mathbb{R}^m)$.

Theorem 1. If $f, p \in \mathcal{F}^T$, $h \in \mathcal{H}$, then for any $t \in [0,T]$, $u^f(\cdot,t) \in \mathcal{H}$ and $u^f \in C([0,T];\mathcal{H})$.

For the inverse problem purposes we assume that $p \in H^1(0,T)$, $p(0) \neq 0$.
The response operator, $R^T : \mathcal{F}^T \to \mathcal{F}^T$, is defined as

$$(R^T f)(t) = u^f(\cdot,t)|_{\partial\Gamma},\ t \in [0,T].$$

Theorem 2. Operator R^T known for any $T > 0$ uniquely determines the graph topology, the lengths of the edges, the potentials q_j and sources h_j, $j = 1,\ldots,N$.

Separating Problem into Two Auxiliary Problems

Solution of the problem (1)–(3) can be presented in the form $u = y + z$:

$$\partial_t y - \partial_x^2 y + q(x)y = 0 \text{ on } E \times (0,T) \tag{4}$$

$$\partial y = f \quad \text{on } \partial\Gamma \times [0,T] \tag{5}$$

$$\partial_t z - \partial_x^2 z + q(x)z = p(t)h(x) \text{ on } E \times (0,T) \tag{6}$$

$$\partial z = 0 \quad \text{on } \partial\Gamma \times [0,T] \tag{7}$$

with the KN matching conditions and zero initial conditions. We see that $z = u^0$ so, $y^f|_{\partial\Gamma} = u^f|_{\partial\Gamma} - u^0|_{\partial\Gamma} = R^T f - R^T 0.$

Solving the Inverse Problem

Prob. 1, (3)-(4), solved previously in **Avdonin and Bell, J. Inv. Prob. and Imaging, 9(3)(2015)**, which dealt with a single unknown conductance $g(x)$, hence no source term. Arguments rely on i) controllability of the system; ii) getting spectral data to an EVP; iii) constructing Boundary Control operators, particularly connection operator C^T, Titchmarch-Weyl (TW) matrix function $\mathcal{M}(\lambda)$; iv) use of a companion inverse problem for the *wave equation* on Γ, to get the graph topology, length of the edges and the potential q are determined on Γ for $T > diam\,\Gamma$.

Prob. 2, (4)-(5), solved in **Avdonin, Bell, and Nurtazina, Math. Meth. Appl. Sci. 40(11)(2017)** utilizing VIE of 2nd kind. Recovery of original distributed parameters requires solving Prob. 1 once, solving Prob. 2 k times.

What Next: Graphs with Cycles

Graphs with cycles

cyclomatic number (first Betti number; rank of fundamental group of Γ considered as a 1D complex) $= |E| - |V| + 1 (> 0)$

Much more challenging extending control theory than with tree graphs

Bioscience applications

- neural circuits of cell clusters (reverberating, parallel, etc.): measuring structural vs functional features
- river delta systems: finding pollution sources, recovering advection parameters
- spider webs: locating the source of vibrations