12 Problems on Semi-infinite Domains and the Laplace Transform

The emphasis up to now has been on problems defined (spatially) on the real line. Now we want to discuss the case of introducing a finite (spatial) boundary so that $(x,t) \in \mathbb{R}^+ \times \mathbb{R}^+$ (i.e., quarter-plane problems). In the sections after this we have our problem defined on bounded spatial domains, so the character of the problems change, and hence the techniques used to solve them.

You have already been exposed to the heat equation on $\mathbb{R}^+ \times \mathbb{R}^+$, where we returned to the use of the general solution $u(x,t) = C_1 \int_0^{x/2\sqrt{Dt}} e^{-r^2} dr + C_2$. Hence, for the problem

$$\begin{cases} \frac{\partial u}{\partial t} = D\frac{\partial^2 u}{\partial x^2} & x > 0, t > 0 \\ u(x,0) \equiv 1 & x > 0 \\ u(0,t) \equiv 0 & t > 0 \end{cases}$$

we determined $u(x,t) = \frac{2}{\sqrt{\pi}} \int_0^{x/2\sqrt{Dt}} e^{-r^2} \ dr = erf(\frac{x}{2\sqrt{Dt}})$. Also, for the problem

$$\begin{cases} \frac{\partial u}{\partial t} = D\frac{\partial^2 u}{\partial x^2} & x > 0, t > 0 \\ u(x,0) \equiv 0 & x > 0 \\ u(0,t) \equiv 1 & t > 0 \end{cases}$$

we determined that $u(x,t) = erfc(\frac{x}{2\sqrt{Dt}}) = 1 - erf(\frac{x}{2\sqrt{Dt}})$. But these boundary conditions are special, given that they are constants. If we modify the last problem slightly, say

$$\begin{cases} \frac{\partial u}{\partial t} = D\frac{\partial^2 u}{\partial x^2} & x > 0, t > 0 \\ u(x,0) \equiv 0 & x > 0 \\ u(0,t) = m(t) & t > 0 \end{cases}$$

where $m(\cdot)$ is an arbitrary, continuous, bounded function, that is *not* constant (the usual physical situation). If you try to apply the general solution form to the heat equation at the boundary $x = 0$, you immediately run into the problem $u(0,t) = C_2 \neq m(t)$ since C_2 is a pure constant. Thus, we have to take a different approach. The Fourier transform is not applicable, but the Laplace transform is.

12.1 A brief review of the Laplace transform

Let $f = f(t)$ be any function defined on $(0, \infty)$, piecewise continuous, and grows at most exponentially; that is, there are positive constants M, a, T such that $|f(t)| \leq Me^{at}$ for all $t > T \geq 0$. Then we define the **Laplace transform** as an operator $\mathcal{L}$ on this set of functions, namely

$$F(s) = \mathcal{L}[f(t)] := \int_0^\infty e^{-st} f(t)\, dt \quad .$$

Just like the Fourier transform we have an inversion formula, a notion of convolution and the convolution theorem, and operational formulas. The inversion formula is given in terms of the contour integral in complex s-space:

$$f(t) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} e^{st} F(s)\, ds = \mathcal{L}^{-1}[F(s)]$$

for a suitably chosen constant c. We will not delve deeply into this topic here, so we just employ a table for inverting expressions from s-space to t-space. For a bit more extensive Laplace transform table, consult a book of tables. Here is part of a typical table most useful to us:

$f(t)$	$F(s) = \int_0^\infty e^{-st} f(t) dt$
t^n	$\frac{n!}{s^{n+1}} \quad (n \geq 0)$
e^{at}	$\frac{1}{s-a}$
$\sin(at)$	$\frac{a}{s^2+a^2}$
$t^{1/2} e^{-a^2/4t}$	$\sqrt{\frac{\pi}{s}}\, e^{-a\sqrt{s}}$
$\frac{a}{2t^{3/2}} e^{-a^2/4t}$	$\sqrt{\pi}\, e^{-a\sqrt{s}}$
$erfc(\frac{a}{2\sqrt{t}})$	$\frac{1}{s} e^{-a\sqrt{s}}$

The convolution of two functions f and g is given by

$$f * g(t) = \int_0^t f(\xi)g(t-\xi)d\xi = \int_0^t f(t-\xi)g(\xi)d\xi \ ,$$

(that is, $f * g(t) = g * f(t)$). The convolution theorem is the same as for the Fourier transform, except for the fact the convolution is defined differently:

$$\mathcal{L}^{-1}[F(s)G(s)] = f * g(t) \ ,$$

where $\mathcal{L}^{-1}[F(s)] = f(t)$, and $\mathcal{L}^{-1}[G(s)] = g(t)$. Besides the obvious linearity of $\mathcal{L}$ and $\mathcal{L}^{-1}$, we have, given $F(s) = \mathcal{L}[f(t)]$,

$$\mathcal{L}[\frac{df}{dt}] = sF(s) - f(0), \mathcal{L}[\frac{d^2f}{dt^2}] = s^2F(s) - sf(0) - f'(0), \text{etc.}$$

Now the **shift formulas**: with $H(t)$ being the Heaviside function,

$$f(t-a)H(t-a) = \mathcal{L}^{-1}[e^{-as}F(s)] \quad (1)$$

$$e^{bt}f(t) = \mathcal{L}^{-1}[F(s-b)] \ . \quad (2)$$

Exercises

1. What is $\mathcal{L}[1]$?
2. $\mathcal{L}[5sinh(2t)]$
3. $\mathcal{L}[e^{at}\cos(kt)]$
4. $\mathcal{L}^{-1}[\frac{1}{s(s^2+1)}]$
5. If $y(t)$ solves $\frac{dy}{dt} + 3y = e^{-t}$, $t > 0$,$y(0) = 1$, what is $Y(s) = \mathcal{L}[y(t)]$?

   Answers: $1/s$, $10/(s^2-4)$, $\frac{s-a}{(s-a)^2+k^2}$, $1-\cos(t)$, $\frac{s+2}{(s+1)(s+3)} = \frac{1}{2}\frac{1}{s+3} + \frac{1}{2}\frac{1}{s+1}$

12.2 Application to heat equation problems

Return to Example 2

$$\begin{cases} \frac{\partial u}{\partial t} = D\frac{\partial^2 u}{\partial x^2} & x > 0, t > 0 \\ u(x,0) \equiv 0 & x > 0 \\ u(0,t) = 1 & t > 0 \end{cases}$$

So, $\mathcal{L}[\frac{\partial u}{\partial t}] = \mathcal{L}[D\frac{\partial^2 u}{\partial x^2}]$, or $sU(x,s) - u(x,0) = DU_{xx}$, where $U(x,s) = \int_0^\infty e^{-st}u(x,t)\,dt$. Then, $U(0,s) = \int_0^\infty e^{-st}\,1\,dt = \mathcal{L}[1] = 1/s$. With

$$U_{xx} - \frac{s}{D}U = 0, \text{ then } U = Ae^{-x\sqrt{s/D}} + Be^{x\sqrt{s/D}}\ .$$

A general condition that is often implicitly assumed for problems with unbounded domains is that u remains bounded as x goes to infinity, for all $t > 0$. (The physics of the problem usually dictates this.) For our example, this means $B = 0$. Thus, $U(x,s) = Ae^{-x\sqrt{s/D}}$ and $1/s = U(0,s) = A$, so $U(x,s) = \frac{1}{s}e^{-x\sqrt{s/D}}$, or $u(x,t) = \mathcal{L}^{-1}[\frac{1}{s}e^{-x\sqrt{s/D}}]$. In the table above ($a = x/\sqrt{D}$), we obtain $u(x,t) = erfc(\frac{x}{2\sqrt{Dt}})$.

Example 3: Same problem, but with $u(0,t) = m(t)$, as given on page 1. Again $U(x,s) = Ae^{-x\sqrt{s/D}}$, but $U(0,s) = \int_0^\infty e^{-st}m(t)dt = M(s)$. So, $U(x,s) = M(s)e^{-x\sqrt{s/D}}$, or $u(x,t) = \mathcal{L}^{-1}[M(s)e^{-x\sqrt{s/D}}]$. Now we have to employ the convolution theorem, since $m(t)$ is arbitrary. Note that

$$\mathcal{L}^{-1}(e^{-x\sqrt{s/D}}) = \frac{x}{2t\sqrt{\pi Dt}}e^{-x^2/4Dt}$$

from the table ($a = x/\sqrt{D}$), so

$$u(x,t) = \frac{x}{2\sqrt{\pi D}}\int_0^t m(t-\tau)\frac{e^{-x^2/4D\tau}}{\tau^{3/2}}\,d\tau\ .$$

The Laplace transform extends our capacity to derive solution formulas, even though we may have to use numerical software to calculate values for u. See Figure 1 showing the graph of $u(x,t)$ for $0 < x < 5$, $0 < t < 10$, when $D = 1$ and $m(t) = \sin(t)$. Note how fast the oscillations created at the boundary damp down as you move away from $x = 0$.

12.3 Digression: Age of the Earth (circa 1860s discussion)

Reference: Arthur Stinner (http://home.cc.umanitoba.ca/ stinner)

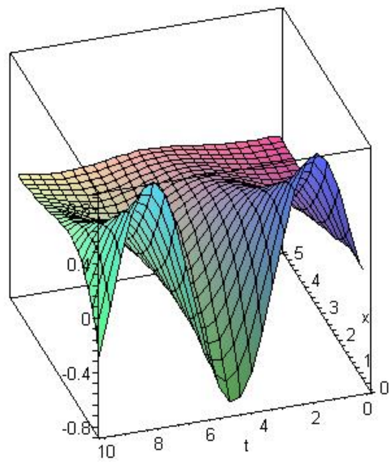

Figure 1: Graph of solution to example 3 with $D = 1$ and $m(t) = \sin(t)$

The issue goes back to antiquity, but from a modeling and computational standpoint, the story will start with William Thomson (Lord Kelvin) in 1862, when he stirred excitement and controversy with the publication of an estimate for the age of the earth. This was done after he had published a similar analysis of the age of the sun. An interesting story is put together by Professor Stinner, a physicist at the University of Manitoba.

Kelvin's main assumptions were: (i) the earth's interior is solid, so only conduction (or heat) is significant; (ii) the earth is basically homogeneous in material properties on a macroscale, and that there is a graded increase in temperature as one goes deeper-this implies a continual loss of heat by conduction; (iii) the earth cooled from about 3700^oC to the present average temperature, which we can take as 0^oC-relative to the previous extremely high temperature, and this is done relatively quickly (40-50 thousand years); (iv) the average temperature of the earth's surface hasn't changed significantly over time. Of course, it is easy to criticize his assumptions. Radioactivity was not discovered yet, let alone the notion of a liquid core (convection processes). For the calculations below we have the half plane (to keep the calculation here to a 1D spatial case), but Kelvin actually treated the earth as a sphere. Hence, with the x-axis pointing downward into the earth (so $x =$ measured depth from the surface), Kelvin employed the problem given in example 2 above, with 1 replaced by $v_0 = 7000^oF$ (so $u(x,t) = v_0 erf(\frac{x}{2\sqrt{Dt}})$) and D

is assumed estimated from measurement of surface rock samples. (Kelvin seemed to make guesses about the melting temperature of rocks; $10,000^oF$ seemed 'unrealistic', and 7000^oF probably seemed 'closer to the truth.' Expanding $erf(z)$ in a Taylor series yields $erf(z) = \frac{2}{\sqrt{\pi}}\{z - z^3/3 + z^5/10 - \ldots\}$, so if we just retain the first term $erf(z) \simeq \frac{2}{\sqrt{\pi}}z$, then $u(x,t) \simeq \frac{v_0 x}{\sqrt{\pi Dt}} := \nu x$ (See Figure 2). If we take a derivative of the actual solution, then

$$\frac{\partial u}{\partial x}(x,t) = \frac{v_0}{\sqrt{\pi Dt}} e^{-x^2/4Dt}$$

or

$$\frac{\partial u}{\partial x}(0,t) = \nu \ ;$$

solving for t gives

$$t = \frac{v_0^2}{\pi D \nu^2} \quad .$$

Therefore, the idea is that a value of ν can come from measurements in deep mines or special deep borings that existed in Kelvin's time. So this gives the temperature gradient that was assumed to be a pretty good estimate of the linear gradient to maybe 30 km into the earth. Hence, values for ν, D, v_0 will give a value for t. Kelvin calculated the temperature gradient of about "1^oF for every 50 feet downward", and with the value of $D \approx 0.012 cm^2/s$ (Edinburgh rocks), then $t \simeq \frac{1}{\pi(0.012cm^2/s)}(\frac{7000^oF}{1^oF/50ft})^2 \simeq \frac{350{,}000^2 ft^2}{\pi(0.012cm^2)} s \simeq 98$ million years. Given the parameter uncertainty, he was willing to consider a longer estimated of 400 million years, but he felt "more comfortable" with the lower estimate. Of course, given what was known at the time, he had no chance to improve his estimate of his controversial calculation. We present this here only to illustrate that there was a little history behind example 2, as simple as it is. (Kelvin's estimate was too short for Darwin's theory of evolution; hence, he became an opponent of Darwin's theory. Even geniuses are wrong at times.)

Exercise: Solve

$$\begin{cases} \frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + V\frac{\partial u}{\partial x} & x > 0, t > 0, \quad V \text{ is a positive constant} \\ u(x,0) = 0 & x > 0 \\ u(0,t) = g(t) & t > 0 \end{cases}$$

Now we introduce a single finite spatial boundary for the wave equation.

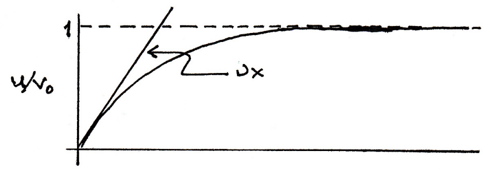

Figure 2: Linear approximation Kelvin used for the error function.

12.4 Application to wave equation problems

Example 4: The semi-infinite vibrating string (or the more colorful name of the *whip-cracking* problem[1]) is

$$\begin{cases} u_{tt} = c^2 u_{xx} & x > 0, t > 0, \quad c > 0 \text{ is a constant} \\ u(x,0) = 0 = u_t(x,0) & x > 0 \\ u(0,t) = m(t) & t > 0 \\ u \text{ remains bounded as } x \to \infty \end{cases}$$

Take the Laplace transform (in t) of the problem:

$$s^2 U - su(x,0) - u_t(x,0) = s^2 U = c^2 U_{xx},$$

and $U(0,t) = \int_0^\infty e^{-st} u(0,t)dt = \int_0^\infty e^{-st} m(t)dt = M(s)$. So $U(x,s) = Ae^{xs/c} + Be^{-xs/c}$, but because of the boundedness condition, $A = 0$. Therefore, $U(x,s) = Be^{-xs/c} = M(s)e^{-xs/c}$.

By the shift formula, $\mathcal{L}^{-1}[e^{-as}F(s)] = H(t-a)f(t-a)$, where in this case $a = x/c$, we have

$$u(x,t) = H(t - x/c)m(t - x/c) = \begin{cases} 0 & \text{if } t - x/c < 0 \\ m(t - x/c) & \text{if } t - x/c \geq 0 \,. \end{cases}$$

Thus, the graph of the solution has the character of that of Figure 3. The solution below the characteristic $t - x/c = 0$ is to be expected to be 0 because any point there has intersecting characteristics that map back to $t = 0$ interval where all the initial data is zero. Note that if $m(0) \neq 0$, then the characteristic $t - x/c = 0$ $(x - ct = 0)$ carries the "corner discontinuity." So the solution has no more smoothness than the boundary data.

Now consider the more general problem

[1] Although, as a colleague pointed out, being an infinitely long string it does not have a chance to actually "crack."

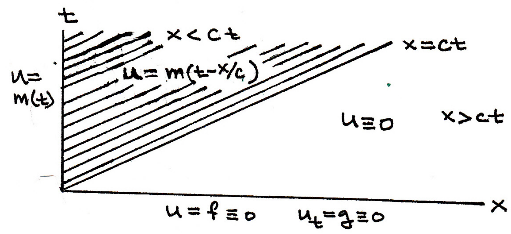

Figure 3: Solution of the whip-cracking problem

$$\begin{cases} u_{tt} = c^2 u_{xx} & x > 0, t > 0, \text{ and } c > 0 \text{ is a constant} \\ u(x,0) = f(x), u_t(x,0) = g(x) & x > 0 \\ u(0,t) = m(t) & t > 0 \end{cases}$$

By linearity we can decompose the problem into two problems, one for $u^{(1)}$ that solves

$$\begin{cases} u_{tt} = c^2 u_{xx} & x > 0, t > 0, \\ u(x,0) = 0, u_t(x,0) = 0 & x > 0 \\ u(0,t) = m(t) & t > 0 \end{cases}$$

and one for $u^{(2)}$ that solves

$$\begin{cases} u_{tt} = c^2 u_{xx} & x > 0, t > 0, \\ u(x,0) = f(x), u_t(x,0) = g(x) & x > 0 \\ u(0,t) = 0 & t > 0 \end{cases}$$

Hence, the solution to the original problem, $u(x,t)$, is given by $u(x,t) = u^{(1)}(x,t) + u^{(2)}(x,t)$. The first problem for $u^{(1)}$ is taken care of by the Laplace transform method (the whipcracking problem), so we concentrate on the second problem. Go back to the general solution of the wave equation, namely,

$$u(x,t) = F(x-ct) + G(x+ct) \; . \tag{3}$$

Recall, when developing the d'Alembert solution we had the following expressions for F and G:

$$F(x) = \tfrac{1}{2} f(x) - \tfrac{1}{2c} \int_0^x g(y) dy + C_1$$

$$G(x) = \tfrac{1}{2} f(x) + \tfrac{1}{2c} \int_0^x g(y) dy + C_2$$

By putting in the correct arguments we arrived at d'Alembert solution

$$u(x,t) = \frac{1}{2}\{f(x-ct) + f(x+ct)\} + \frac{1}{2c} \int_{x-ct}^{x+ct} g(y) dy \; .$$

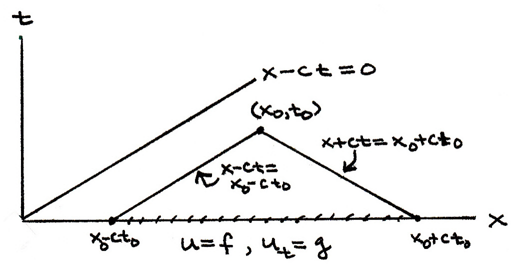

Figure 4: For (x_0, t_0) with $x_0 - ct_0 > 0$.

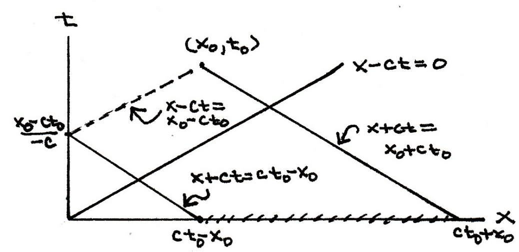

Figure 5: For (x_0, t_0) with $x_0 - ct_0 < 0$.

A problem arises here because in the problem above f and g are only defined for positive argument, so $F(x)$ is only defined for positive argument. Hence, we can only use $F(x - ct)$ for $x - ct > 0$. But, as x and t range over $(0, \infty)$, $x - ct$ ranges over $(-\infty, \infty)$. We need to know F for negative values of its argument! There is no problem with G since $x, t > 0$ implies $x + ct > 0$ (recall: $c > 0$). The point is to apply the boundary condition $u^{(2)}(0, t) = 0$: at $x = 0$

$$0 = F(-ct) + G(ct) \qquad \text{from (3)}$$

or, writing $y = -ct$, $F(y) = -G(-y)$ for $y < 0$, which gives F for negative argument since the right-hand side is known. Thus, for $0 < x < ct$ $(x - ct < 0)$

$$F(x - ct) = -G(ct - x) = -\frac{1}{2}f(ct - x) - \frac{1}{2c}\int_0^{ct-x} g(y)dy - C_2\ .$$

Since d'Alembert's formula is ok for $x - ct > 0$, we have

$$u^{(2)}(x,t) = \begin{cases} \frac{1}{2}\{f(x-ct) + f(x+ct)\} + \frac{1}{2c}\int_{x-ct}^{x+ct} g(y)dy & x > ct \\ \\ \frac{1}{2}\{f(ct+x) - f(ct-x)\} + \frac{1}{2c}\int_{ct-x}^{ct+x} g(y)dy & x < ct \end{cases} \tag{4}$$

A pictorial interpretation of (4) is given in Figures 4-5 below.

Summary: Given a Laplace transform table, know how to use the transform to solve problems. Although you need not memorize the d'Alembert formula (4) for the quarter-plane problem, be able to refer back to it and understand the geometric meaning as depicted in Figures 4-5.

Exercises

1. Find the bounded solution to the wave equation $u_{tt} = 4u_{xx}$, $x > 0, t > 0$, with the following boundary conditions:

   (a) $u(x, 0) = \cos(x)$, $u_t(x, 0) = 0$, for $x > 0$, and $u(0, t) = 1$, for $t > 0$.

   (b) $u(x, 0) = 0$, $u_t(x, 0) = \cos(x)$, $x > 0$, and $u(0, t) = 0$, for $t > 0$.

   (c) $u(x, 0) = e^{-x^2}$, $u_t(x, 0) = 1$, $x > 0$, and $u(0, t) = 0$ for $t > 0$.

2. Consider the initial-boundary value problem [2]

$$\begin{cases} u_t = u_{xx} & x > 0, t > 0 \\ u(x,0) = f(x) & x > 0 \\ u_x(0,t) - u(0,t) = 0 & t > 0, \end{cases}$$

   and u remains bounded. Solve this problem by observing that the function $v = u_x - u$ satisfies the problem

$$\begin{cases} v_t = v_{xx} & x > 0, t > 0 \\ v(x,0) = f'(x) - f(x) & x > 0 \\ v(0,t) = 0 & t > 0 \end{cases}$$

   Let $f(x) = e^{-x}$.

3. Consider the problem

$$\begin{cases} u_{tt} = u_{xx} & x > 0, t > 0 \\ u(x,0) = 0 = u_t(x,0) & x > 0 \\ u_x(0,t) - hu(0,t) = g(t) & t > 0 \end{cases}$$

   where $h > 0$ is a constant, $g(t)$ is any continuous function with a Laplace transform. Also assume $u \to 0$ as $x \to \infty$ for all $t > 0$.

[2] This problem suggested itself when the author was looking at a bread-baking modeling problem

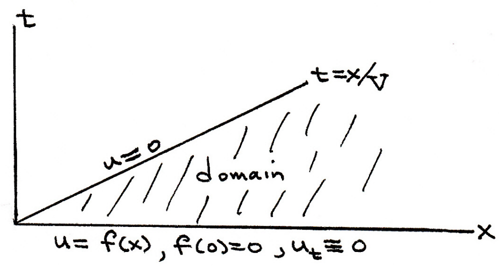

Figure 6: For problem 4.

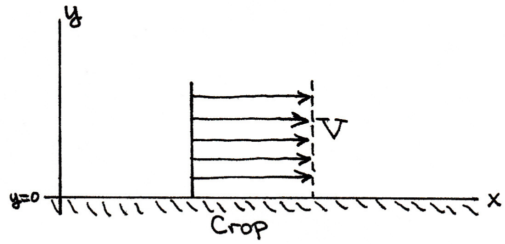

Figure 7: For problem 5.

(a) What is the domain of dependence for the point $(x,t) = (6,4)$?

(b) Use the Laplace transform in t, $U(x,s) = \int_0^\infty e^{-st}u(x,t)dt$, to solve the problem.

4. Consider the problem

$$\begin{cases} u_{tt} = c^2 u_{xx} & x > Vt, t > 0 \\ u(x,0) = f(x), u_t(x,0) = g(x) & x > 0 \\ u(Vt,t) = 0 & t > 0 \\ c, V > 0 \text{ are constants} & \end{cases}$$

Solve when

(a) $V > c$ (the supersonic case)

(b) $0 < V < c$ (the subsonic case)

Interpret your results. (See Figure 6.)

5. A field of crops has been sprayed with a pesticide. The wind picks up some of the pesticide and carries it along. There is also diffusion of the

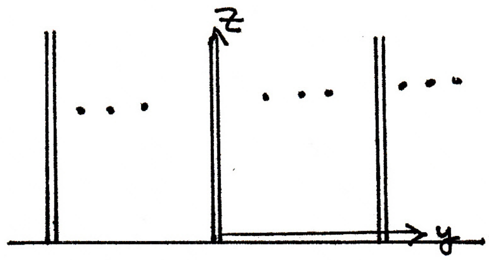

Figure 8: For problem 6.

pesticide caused by turbulent fluctuations mainly in the y direction. Let V be the constant wind velocity (see Figure 7), and let $c = c(x, y)$ be the steady state (i.e. time independent) concentration of the pesticide at height y above the crop, and distance x from the start of the field. A model for this situation is

$$V\frac{\partial c}{\partial x} = D\frac{\partial^2 c}{\partial y^2} \quad x > 0, y > 0$$
$$c(x, 0) \equiv c_0 = \text{constant} > 0, \quad c(0, y) = 0 \quad .$$

Solve this air-born pesticide concentration problem in terms of error functions.
(answer: $c(x, y) = c_0 erfc(y\sqrt{V/Dx}/2)$)

6. A rough model of a paint brush is to consider the brush being composed of a large number of parallel and equal-distant thin, rigid plates which slide together over a plane wall[3] (see Figure 8). Space between the plate (infinite in the x-direction) is filled with liquid and the plates are drawn across the wall in unidirectional, steady motion. A no-pressure gradient is imposed, so the governing equation is $u_{yy} + u_{zz} = 0$, $z > 0$. It is convenient to take the axes between the plates, and therefore have the boundary conditions $u(0, z) = 0 = u(b, z), z \in (0, \infty)$, while $u(y, 0) = U$ = relative speed of the brush, $0 < y < b$. Solve the problem.

[3]This problem was motivated from discussion of the topic in G. Batchelor's book *An Introduction to Fluid Dynamics.*