7 Transmission Line Equation (Telegrapher's Equation) and Wave Equations of Higher Dimension

7.1 Telegrapher's equation

Consider a piece of wire being modeled as an electrical circuit element (see Figure 1) consisting of an infinitesimal piece of (telegraph) wire of resistance $R\triangle x$ and inductance $L\triangle x$, while it is connected to a ground with conductance $(G\triangle x)^{-1}$ and capacitance $C\triangle x$. Let $i(x,t)$ and $v(x,t)$ denote the current and voltage across the piece of wire at position x at time t. The change in voltage across the piece of wire is given by $v(x+\triangle x,t)-v(x,t) = -i(x,t)R\triangle x - \frac{\partial i}{\partial t}(x,t)L\triangle x$, and the amount of current that disappears via the ground is $i(x+\triangle x,t)-i(x,t) = -i(x,t)G\triangle x - \frac{\partial v}{\partial t}(x,t)C\triangle x$. Dividing by $\triangle x$ and letting $\triangle x \to 0$ gives

$$\frac{\partial v}{\partial x} = -Ri - L\frac{\partial i}{\partial t}\,, \quad \frac{\partial i}{\partial x} = -Gv - C\frac{\partial v}{\partial t} \quad .$$

Eliminating i to combine the equations gives

$$LC\frac{\partial^2 v}{\partial t^2} + (LG+RC)\frac{\partial v}{\partial t} + RGv = \frac{\partial^2 v}{\partial x^2} \quad . \tag{1}$$

If we define $c := 1/\sqrt{LC}$, $a := c^2(LG+RC)$, $b := c^2RG$, then (1) becomes

$$\frac{\partial^2 v}{\partial t^2} + a\frac{\partial v}{\partial t} + bv = c^2\frac{\partial^2 v}{\partial x^2}. \tag{2}$$

This equation, or (1), is referred to as the **telegrapher's equation**. For reasons we will explain below the $a\partial v/\partial t$ term is called the *dissipation term*, and the bv term is the *dispersion term*.

Of course, if $a = b = 0$, we are back to the vibrating string, i.e. wave equation, with its right and left moving wave solution representation. A natural question to ask is: Can we make a change of variables to reduce (2) to the wave equation? Let's see. If we let $v(x,t) = w(t)u(x,t)$, the idea is to pick $w(t)$ to obtain a "reduced" equation for u. Substituting this form into (2) gives

$$wu_{tt} + \{2\frac{dw}{dt} + aw\}u_t + \{\frac{d^2w}{dt^2} + a\frac{dw}{dt} + bw\}u = c^2wu_{xx} \quad .$$

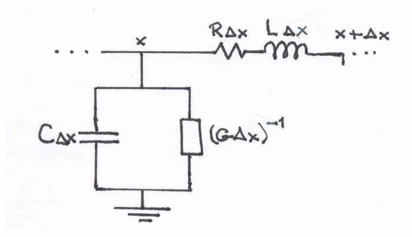

Figure 1: Circuit model element for the telegraph equation

You could try to set both bracketed expressions to zero, but you will find that this is only possible if $b - a^2/4 = 0$. Assume temporarily that $b \neq a^2/4$ so that both expressions can not simultaneously be zero. We can set the first bracketed expression equal to zero and solve the first-order equation, or set the second bracketed expression equal to zero and solve the second-order equation. Let us do the former calculation: $\frac{dw}{dt} + \frac{1}{2}aw = 0$ to eliminate the u_t term. This gives, up to an irrelevant multiplicative factor, $w(t) = e^{-at/2}$. If we define $k := b - a^2/4$, then

$$u_{tt} + ku = c^2 u_{xx} \quad . \tag{3}$$

This is not the wave equation. Consider first the case when $k = 0$, i.e. $b = a^2/4$. Then u satisfies the wave equation, so the general solution for the v equation is

$$v(x,t) = w(t)u(x,t) = e^{-at/2}\{F(x-ct) + G(x+ct)\} \quad .$$

Thus, the right and left moving waves would retain their shape (given by F and G) except now there is an *amplitude-attenuation factor* that depends on time. That is, the waves remain relatively undisturbed. The attenuation requires extra energy being put into the system periodically, but it is a desirable property to have the waves retain their shape if it is carrying specific information. By returning to the original variables, $b = a^2/4$ implies $4(LC)(RG) = (RC + LG)^2 = (RC)^2 + 2(LC)(RG) + (LG)^2$ if, and only if $0 = (RC - LG)^2$; so the circuit parameters needed to have the dispersionless case $k = 0$ is for $RC = LG$. Since the attenuation factor involves a, that is one reason for calling $a\partial v/\partial t$ a dissipation term.

Remark: As a historical aside, William Thomson (later Lord Kelvin), the great 19th century mathematical physicist, was very instrumental in the British effort to lay the trans-Atlantic telegraph cable, an effort started in 1858. Kirchhoff was probably the first to write down the telegraph equation, but Thomson certainly had done some analysis on it to draw the conclusions he did. Oliver Heaviside sometime later also wrote down the equation, and maybe the first to realize that physical constants could be adjusted to eliminate the dispersion. But the means to do this went to Michael Pupin, a Serbian born American engineer, so the above case corresponds to what became known as "Pupinizing" the cable. Our interest in this section in introducing the telegraph equation is to see what comes out of the adding of lower order terms to the wave equation.

In case $k \neq 0$, (3) is not a wave equation so we will defer a full discussion of the solution here. But we examine when (3) has *progressive wave* solutions. Let $u(x,t) = \phi(x - ct) = \phi(z)$. Substituting this into (3) gives

$$c^2 \frac{d^2\phi}{dz^2} + k\phi = c^2 \frac{d^2\phi}{dz^2} \Rightarrow \phi \equiv 0 \ .$$

So there is no nontrivial solution of this form. Letting $u = \phi(x + ct)$ does not give a non-zero solution either since the argument does not depend on the sign of c. Now try $u(x,t) = \phi(x - \gamma t) = \phi(z)$, $\gamma \neq \pm c$. Then, upon substituting into (3),

$$(\gamma^2 - c^2)\frac{d^2\phi}{dz^2} + k\phi = 0 \quad \text{or} \quad \frac{d^2\phi}{dz^2} + \mu^2\phi = 0 \ ,$$

where $\mu^2 = k/(\gamma^2 - c^2)$. So, we have bounded, oscillatory wave solutions to (3) for **every** speed γ, with $|\gamma| > |c|$. Any sum of these wave solutions is a wave solution, and waves that do not propagate at the same speed are **dispersive**; hence, (3) is a **dispersive hyperbolic equation**.

Exercises

1. If you let $v(x,t) = e^{\alpha x - \beta t}u(x,t)$ instead of the above form, show that $\alpha^2 = k$, k as given above. What is β and what is the resulting equation for u?

2. If we define $E(t) = \frac{1}{2}\int_{-\infty}^{\infty}\{c^{-2}u_t^2 + u_x^2 + ku^2\}dx$ for some smooth solution to (3) on the real line, then show that $E(t)$ is independent of t. Can you draw any conclusions from this?

Remark: The equation (3) is also a linear *Klein-Gordon* equation associated with quantum mechanics (used to describe a "scalar" meson if k is taken as m^2, where m is mass of the particle). It can also be thought of as a model for a flexible string with additional stiffness provided by the surrounding medium. The equation is not only of dispersive type, but is also conservative (see the above exercise).

Remark: If there is no inductance in the transmission line, then $L = 0$ in (1), so $RC\frac{\partial v}{\partial t} + RGv = \frac{\partial^2 v}{\partial x^2}$, or more commonly,

$$C\frac{\partial v}{\partial t} + Gv = \frac{1}{R}\frac{\partial^2 v}{\partial x^2}. \tag{4}$$

Note that (4) is a diffusion equation, not a wave equation. What "transmission line" has no inductance? Well, axons and dendrites of nerve cells. A reason for this is that the carrier of current are ions, not electrons. In this situation (4) is called a (linear) cable equation[1]. Many small, short dendrites are considered linear cables, so (4) is a reasonable description for the dynamics of transmembrane potential $v(x, t)$. But larger dendrites and all axons must carry discrete signals (propagated action potentials) a considerable distance, so an adequate description of this signal propagation must replace the linear term Gv with an expression that is *nonlinear* in $v(x, t)$. Wave solutions of the above type is an important concept in nonlinear PDEs too.

7.2 Plane waves and the dispersion relation

Wave solutions are a central idea in engineering and the physical sciences, so we need a bit more terminology. For linear equations we look for solutions of the form (in one space dimension) $u(x, t) = A\cos(kx - \omega t)$, where A is the amplitude, k is the wave number (measure of the number of spatial oscillations per 2π space units, observed at a fixed time), ω is the frequency (a measure of the number of oscillations in time per 2π units, observed at a

[1]Apparently this is the way Thomson first viewed the telegrapher's equation. Discussion on this can be found in Jeremy Gray's *Henri Poincare: A Scientific Biography*, Princeton University Press, 2013

fixed spatial location). Other notable quantities often discussed are $\lambda = 2\pi/k$ = wavelength (distance between peaks), $p = 2\pi/\omega$ = period (time scale of repeated pattern), and $c = \omega/k$ = phase velocity (speed one has to move to keep up with a wave crest).

For calculating purposes, instead of the above form we use $u(x,t) = Ae^{i(kx-\omega t)}$ in the equation, then take real or imaginary parts when necessary.

Example: the heat equation $u_t = Du_{xx}$
Upon substitution of the $u = Ae^{i(kx-\omega t)}$ into the heat equation we obtain

$$-i\omega Ae^{i(kx-\omega t)} = (ik)^2 DAe^{i(kx-\omega t)} \quad \Rightarrow \quad \omega = -ik^2 D \ .$$

The relationship between frequency and wavenumber, $\omega = \omega(k)$, is called a **dispersion relation**. Note that if we substitute this relation back into the form of u, we have $u(x,t) = \left(Ae^{-Dk^2 t}\right)\left(e^{ikx}\right)$. We have written this as a product of two quantities, namely dissipation term times a spatial oscillation term. So the rate of decay of a plane wave depends on the wavenumber; waves of shorter wavelength (larger wavenumber) decay more rapidly than waves of longer wavelength.

Example: the wave equation $u_{tt} = c^2 u_{xx}$
Upon substitution of $u(x,t) = Ae^{i(kx-\omega t)}$, we obtain

$$(-i\omega)^2 = c^2(ik)^2 \quad \Rightarrow \quad \omega^2 = c^2k^2 \quad \Rightarrow \quad \omega = \pm ck \ .$$

Thus, putting this dispersion relation back into the plane wave solution, we have $u(x,t) = Ae^{-k(x\pm ct)}$, which give the right and left moving sinusoidal traveling waves of speed c.

Definition: An equation is **dispersive** if $\omega(k)$ is real and $d^2\omega/dk^2(k) \neq 0$. If $\omega(k)$ is complex, the equation is **diffusive**.
Dispersion relations are sometimes used to classify equations, and the concept carries over to higher dimensional equations and nonlinear equations also. Note that by this definition, the heat equation is diffusive, but the wave equation is **not** dispersive.

Example: Klein-Gordon equation $u_{tt} + m^2 u = c^2 u_{xx}$
Again substituting the plane wave solution representation, we obtain

$$(i\omega)^2 + m^2 = c^2(ik)^2 \quad \Rightarrow \quad \omega = \pm\sqrt{c^2k^2 + m^2},$$

which makes the Klein-Gordon equation dispersive, consistent with our discussion in the previous subsection.

Exercises:
For the following equations, find the dispersion relation and classify the equation as diffusive, dispersive, or neither:

1. $u_{tt} + a^2 u_{xxxx} = 0$ (beam equation)

2. $u_t + au_x + bu_{xxx} = 0$ (linear Korteweg-deVries equation)

3. $u_t = iu_{xx}$ (*free* Schrödinger equation)

4. $u_t + u_{xxx} = 0$ (Airy's equation)

5. $u_{tt} = c^2 u_{xx} + du_{xxt}$ (String equation with Kelvin-Voigt damping)

Remark: Plane wave solutions: For higher space dimensions, the wave number becomes a vector of wave numbers in each direction, so plane wave solutions take the form $u(\mathbf{x}, t) = Ae^{i(\mathbf{k}\cdot\mathbf{x}-\omega t)}$. Using this in the multidimensional version of Schrödinger equation, $u_t = i\nabla^2 u$, gives $\omega = -|\mathbf{k}|^2 = -\sum_{i=1}^{n} k_i^2$. Then the condition for an equation to be dispersive is for the determinant of $W = (\partial^2\omega/\partial k_i \partial k_j) \neq 0$.

Remark: Returning to the plane wave solution of the diffusion equation, we can think of having a one-parameter family of solutions, one for each wavenumber with not necessarily the same amplitude:

$u(x, t; k) = A(k)e^{-Dk^2 t}e^{ikx}$, $k \in \mathbb{R}$.

We can formally superimpose such solutions to make another solution, and running over all possible cases gives

$$u(x, t) = \int_{-\infty}^{\infty} A(k)e^{-Dk^2 t}e^{ikx}\, dk \quad ,$$

where we assume the amplitude is k-dependent and well-behaved, say $A(k)$ is continuous, bounded, absolutely integrable. Then

$$u(x, 0) := f(x) = \int_{-\infty}^{\infty} A(k)e^{ikx}\, dk$$

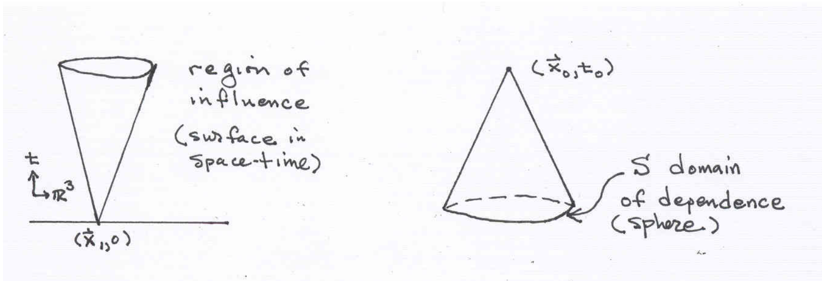

Figure 2: Light cone in $\mathbb{R}^3$

is the Fourier transform of $A(k)$: that is, $A(k)$ is the **inverse Fourier transform** of $f(x)$. We will discuss Fourier transforms in Section 12.

7.3 Wave equation in higher dimensions

Consider the Cauchy problem in three space

$$\begin{cases} \frac{\partial^2 u}{\partial t^2} = c^2 \nabla^2 u & \mathbf{x} \in \mathbb{R}^3, t > 0 \\ u(\mathbf{x}, t) = f(\mathbf{x}), \frac{\partial u}{\partial t}(\mathbf{x}, 0) = g(\mathbf{x}) & . \end{cases} \tag{5}$$

The (compact) formula for the solution to (5), analogous to d'Alembert's formula, is

$$u(\mathbf{x}, t) = \frac{1}{4\pi c^2 t} \int_S g(\mathbf{x}') d\mathbf{x}' + \frac{\partial}{\partial t} \{ \frac{1}{4\pi c^2 t} \int_S f(\mathbf{x}') d\mathbf{x}' \} \tag{6}$$

where $S = S(\mathbf{x}, t)$ is the sphere centered at $\mathbf{x}$ with radius ct. (This formula is due to Poisson, but is known as *Kirchhoff's formula*.)

So the value of $u(\mathbf{x}, t)$ depends, from (6), just on the values of $f(\mathbf{z})$ and $g(\mathbf{z})$ for $\mathbf{z}$ on the spherical surface $S(\mathbf{x}, t) = \{\mathbf{z} \in \mathbb{R}^3 : |\mathbf{z} - \mathbf{x}| = ct\}$, but **not** on the values of f and g inside the sphere. Another way of interpreting this is to say that the values of f and g at a point $\mathbf{x}$, influence the solution on the surface $\{|\mathbf{x} - \mathbf{x}_1| = ct\}$ of the *light cone* that emanates from $(\mathbf{x}_1, 0)$. (See Figure 2.)

This observation relates to **Huygen's principle**. That is, any solution of the 3D wave equation (e.g. any electromagnetic signal in a vacuum) propagates at *exactly* the speed c of light, no faster and no slower. This principle allows us to see sharp images. It means that any sound is carried through the air at exactly a fixed speed and without "echoes" (density and velocity of small acoustic disturbances follow the wave equation), assuming the absence of walls or inhomogeneities in the air. Thus, at any time t a listener hears exactly what has been played at the time $t - d/c$, where d is the distance to the source (musical instrument, for example), rather than a mixture of the notes played at various earlier times.

Now consider the Cauchy problem in 2D:

$$\begin{cases} \frac{\partial^2 u}{\partial t^2} = c^2 \nabla^2 u & \mathbf{x} = (x_1, x_2) \in \mathbb{R}^2, t > 0 \\ u(\mathbf{x}, t) = f(\mathbf{x}), \frac{\partial u}{\partial t}(\mathbf{x}, 0) = g(\mathbf{x}). \end{cases} \tag{7}$$

The analogue to Kirchhoff's formula is

$$u(\mathbf{x}, t) = u(x_1, x_2, t) = \begin{cases} \frac{1}{2\pi c} \int_D \frac{g(y_1, y_2) dy_1 dy_2}{\sqrt{c^2t^2 - (x_1 - y_1)^2 - (x_2 - y_2)^2}} + \\ \frac{\partial}{\partial t} \{ \frac{1}{2\pi c} \int_D \frac{f(y_1, y_2) dy_1 dy_2}{\sqrt{c^2t^2 - (x_1 - y_1)^2 - (x_2 - y_2)^2}} \end{cases}, \tag{8}$$

where $D = D(\mathbf{x}, t)$ is the disk $\{(y_1, y_2) : (x_1 - y_1)^2 + (x_2 - y_2)^2 \leq c^2t^2\}$. Thus, formula (8) shows that the value of $u(\mathbf{x}, t)$ depends on the values of $f(\mathbf{z})$ and $g(\mathbf{z})$ **inside** the cone

$$(x_1 - y_1)^2 + (x_2 - y_2)^2 \leq c^2t^2.$$

Communication would be a nightmare because sound and light waves would not propagate sharply. It would be very noisy because of all the echoing. So living in 3D (space) is a good thing.

Remark: A restatement of Huygen's principle is that in *odd* dimensions greater than one we get sharp signals from a point source, but in *even* space dimensions this is violated. Figure 3 illustrates Huygen's principle for dimension one.

Summary: Be able to find the dispersion relation for a given equation and know how to classify equations based on it.

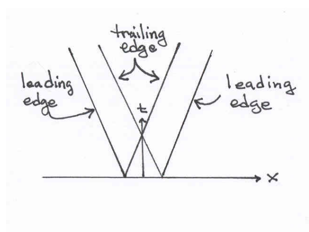

Figure 3: Huygen's principle in one space dimension